Section 4.1 - Review

Differentials - Recall

recall: $\frac{dy}{dx} = f'(x)$ derivative

$\longrightarrow dy = f'(x)\,dx$ differential

$(ax^n)' = nax^{n-1}$

ex. find differential for $\frac{d}{dx}\left(y = (\overbrace{1+x^3}^{u})^{-2}\right)$

$\frac{dy}{dx} = -2(1+x^3)^{-3} \cdot 3x^2$

$\rightarrow dy = \frac{-6x^2}{(1+x^3)^3}\,dx$

Section 5.5 - Substitution Rule

Use Substitution Rule when there are 2 $\overbrace{\text{functions}}^{f(x)}$ within the integration AND one is the derivative of the other.

indefinite integral

ex. calculate $\int 2x\sqrt{1+x^2}\,dx$

$(1+x^2)' = 2x$

$u = 1+x^2$

take differential $du = 2x\,dx$

reorder $= \int \sqrt{1+x^2}\; 2x\,dx$

rewrite ITO u $= \int \sqrt{u}\;du$

$= \int u^{1/2}\,du$

$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C$ where $n \neq -1$

$= \frac{2}{3}u^{3/2} + C$

$\frac{u^{1/2+1}}{1/2+1} = \frac{u^{3/2}}{3/2} \xrightarrow{KCF} \frac{2}{3}u^{3/2}$

substitute back for x $= \boxed{\frac{2}{3}(1+x^2)^{3/2} + C}$

Do: check by taking derivative

$\left(\frac{2}{3}(1+x^2)^{3/2} + C\right)'$

$\cancel{\frac{2}{3}}\cancel{\frac{3}{2}}(1+x^2)^{1/2}\,2x + 0$

$\sqrt{1+x^2}\,2x$ ☺

$(ax^n)' = nax^{n-1}$

$\int c \cdot f(x)dx = c\int f(x)dx$

Examples when u must be manipulated:

ex. evaluate $\int (x^3+1)^5\, 3x^2\,dx$

$u = x^3+1$

$du = 3x^2\,dx$

$\longrightarrow = \int u^5\,du$

ex find $\int (x^3+1)^5\, x^2\,dx$

$u = x^3+1$

$\frac{1}{3}du = \frac{\cancel{3}}{\cancel{3}}x^2\,dx$

$$\longrightarrow = \int u^5\,du \qquad \underline{du} = 3x^2\,dx$$
$$= \frac{u^6}{6} + C$$
$$= \boxed{\frac{(x^3+1)^6}{6} + C}$$

$u = x^3+1$

$\frac{1}{3}du = \frac{3x^2}{3}dx$

$\frac{1}{3}du = x^2dx$

$$= \int u^5 \cdot \frac{1}{3}\,du$$
$$= \frac{1}{3}\int u^5\,du$$
$$= \frac{1}{3}\cdot\frac{u^6}{6} + C$$
$$= \frac{u^6}{18} + C$$
$$= \boxed{\frac{(x^3+1)^6}{18} + C}$$

ex. evaluate $\int \frac{x}{\sqrt{1-4x^2}}\,dx \xrightarrow{\text{rewrite}} \int \frac{1}{\sqrt{1-4x^2}}\,x\,dx$

$u = 1-4x^2$

$-\frac{1}{8}du = \frac{-8x}{-8}dx$

$-\frac{1}{8}du = x\,dx \leftarrow$

$$= \int \left(\frac{1}{\sqrt{u}}\right)\left(-\frac{1}{8}du\right)$$
$$= -\frac{1}{8}\int u^{-1/2}\,du$$
$$= -\frac{1}{\cancel{8}\,4}\cancel{2}\,u^{1/2} + C$$
$$= -\frac{\sqrt{1-4x^2}}{4} + C \xleftarrow{\text{same}} -\frac{1}{4}\sqrt{1-4x^2} + C$$

ex. calculate $\int e^{5x}\,dx$

$u = 5x$

$du = 5dx$

$\frac{1}{5}du = dx$

$$= \int e^u\left(\frac{1}{5}du\right)$$
$$= \frac{1}{5}\int e^u\,du$$
$$= \frac{1}{5}e^u + C$$
$$= \frac{1}{5}e^{5x} + C \quad \text{or} \quad \frac{e^{5x}}{5} + C$$

Do: $\int e^{x/2}\,dx = \boxed{2e^{x/2} + C}$

$$= 2\int e^u\,du$$
$$= 2e^u + C$$
$$= 2e^{x/2} + C$$

$u = \frac{1}{2}x$

$du = \frac{1}{2}dx$

$2\,du = dx$

Do: $\int \frac{x}{1-6x^2}\,dx = \boxed{-\frac{1}{12}\ln|1-6x^2| + C}$

$$= \int \frac{1}{1-6x^2}\,x\,dx$$
$$= -\frac{1}{12}\int \frac{1}{u}\,du$$
$$= -\frac{1}{12}\ln|u| + C \qquad (\ln u)' = \frac{1}{u}$$

$u = 1-6x^2$

$-\frac{1}{12}\cdot du = \frac{-12x\,dx}{-12}$

$$= -\frac{1}{12}\ln\left|1-6x^2\right| + C$$

ex. calculate $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$

$$= \int \frac{1}{\cos x} \cdot \sin x\, dx$$

$$= -\int \frac{1}{u}\, du$$

$$= -\ln|u| + C$$

$$= \boxed{-\ln|\cos x| + C}$$

$u = \cos x$

$-du = +\sin x\, dx$

Do: calculate $\int \cot x \csc x\, dx$

*HINT: rewrite trig functions ITO sine and cosine

$$= \int \frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}\, dx$$

$$= \int \frac{1}{\sin^2 x}\, \underbrace{\cos x\, dx}_{du}$$

$\frac{1}{(\sin x)^2}$

$$= \int \frac{1}{u^2}\, du$$

$$= \int u^{-2}\, du$$

$$= \frac{u^{-1}}{-1} + C$$

$$= -\frac{1}{u} + C \longrightarrow \boxed{-\frac{1}{\sin x} + C}$$

$u = \sin x$

$du = \cos x\, dx$

$\int \sqrt{u} \cdot \frac{1}{2} du$

ex. calculate $\int \sqrt{2x+1}\, dx$

$$= \frac{1}{2}\int \sqrt{u} \cdot du$$

$$= \frac{1}{2}\int u^{1/2}\, du$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{3} u^{3/2} + C$$

$$= \frac{1}{3}(2x+1)^{3/2} + C$$

$u = 2x+1$

$du = 2\, dx$

$\frac{1}{2} du = dx$

Substitution Rule for Definite Integrals

If g' is continuous on $[a,b]$ and f is continuous on the range where $u=g(x)$ then $\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

$$= F(u)\Big|_{g(a)}^{g(b)}$$

$$= F(g(b)) - F(g(a))$$

i.e. bounds must reflect variable being used for integration

re-using $f(x)=\sqrt{2x+1}$:

ex. evaluate $\int_0^4 \sqrt{2x+1}\,dx$

$$= \frac{1}{2}\int_{0=x}^{4=x} \sqrt{u}\,du$$

$$= \frac{1}{2}\int_{+1}^{9} u^{1/2}\,du$$

$$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\,u^{3/2}\Big|_1^9 \to \frac{1}{3}\left(9^{3/2} - 1^{3/2}\right) = \frac{1}{3}(27-1) = \frac{1}{3}\cdot 26 = \boxed{\frac{26}{3}}$$

F(b) - F(a)

$u = 2x+1$
$du = 2dx \Rightarrow \frac{1}{2}du = dx$

$u_{x=0}: 2(0)+1 = +1$
$u_{x=4}: 2(4)+1 = 9$

ex. evaluate $\int_1^2 \frac{dx}{(3-5x)^2} = \int_1^2 \frac{1}{(3-5x)^2}\,dx$

$$= -\frac{1}{5}\int_{-2}^{-7} \frac{1}{u^2}\,du$$

Do: $= -\frac{1}{5}\cdot\frac{u^{-1}}{-1}\Big|_{-2}^{-7}$

$$= \frac{1}{5}\cdot\frac{1}{u}\Big|_{-2}^{-7}$$

$$= \frac{1}{5}\left(\frac{1}{-7} - \frac{1}{-2}\right)$$

F(b) - F(a)

$$= \frac{1}{5}\left(\frac{7}{7}\cdot\frac{1}{2} - \frac{1}{7}\cdot\frac{2}{2}\right)$$

$$= \frac{1}{\cancel{5}}\cdot\frac{\cancel{5}}{14} = \boxed{\frac{1}{14}}$$

$u = 3-5x$
$du = -5\,dx \Rightarrow -\frac{1}{5}du = dx$

$u_{x=1}: 3-5(1) = -2$
$u_{x=2}: 3-5(2) = -7$

ex. evaluate $\int_1^e \frac{\ln x}{x}\,dx = \int_1^e \ln x\cdot\frac{1}{x}\,dx$

$$= \int_0^1 u\cdot du$$

$u = \ln x$
$du = \frac{1}{x}\,dx$

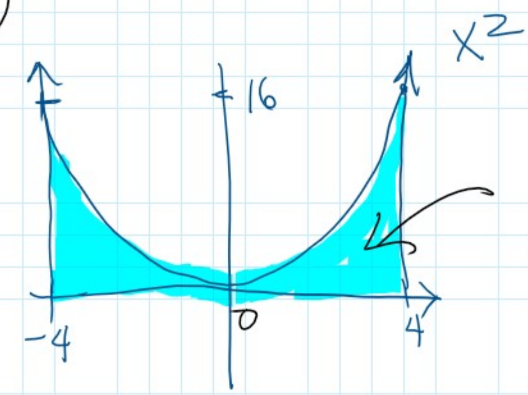
$u_{x=1}: \ln 1 = 0$

$$= \int_0^1 u \cdot du \qquad \hookrightarrow u_{x=1}: \ln 1 = 0 \qquad u_{x=e}: \ln e = 1$$

$$= \frac{1}{2} u^2 \Big|_0^1$$

$$= \frac{1}{2}\left(\cancel{1^2}^{\,1} - \cancel{0^2}^{\,0}\right)$$

$$= \boxed{\frac{1}{2}}$$



SYMMETRY - REVIEW

see examples in slide pack

$\boxed{\text{EVEN FUNCTIONS}}$ given $f(x)$, $f(x)$ is even when $f(-x) = f(x)$ for all x in domain of f

ex. of even functions

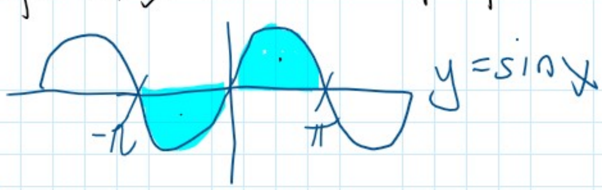
x^2, $\cos x$ $\qquad f(-x) = (-x)^2 = x^2 = f(x)$.

even functions have y-axis symmetry

ODD FUNCTIONS given $f(x)$ $f(x)$ is odd when $\boxed{f(-x) = -f(x)}$ for all x in domain of f

ex. x^3, $\sin x$, $\tan x$

odd functions have origin symmetry



Do: determine if following functions are even or odd.

$f(x) = x^4 + 1$
even

$f(x) = \frac{\tan x}{x^2}$ $\qquad f(-x) = \frac{\tan(-x)}{(-x)^2} = \frac{-\tan x}{x^2} = -\frac{\tan x}{x^2} = -f(x)$

$\therefore f(x)$ is odd

Integrals of Symmetric Functions

Rules

suppose f is continuous on $[-a, a]$

if f is even: $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

if f is odd: $\int_{-a}^{a} f(x)\,dx = 0$

ex. $\int_{-1}^{1} (x^4+1)\,dx \xrightarrow[\text{is even}]{x^4+1} = 2\int_0^1 (x^4+1)\,dx$

$f(-x) = \cdots$

$$= 2\left(\frac{x^5}{5} + x\right)\Big|_0^1 \longrightarrow 2\left(\overbrace{\left(\frac{1^5}{5} + 1\right)}^{F(b)} - \overbrace{\left(\frac{0^5}{5} + 0\right)}^{F(a)}\right)$$

$$= 2\left(\frac{1}{5}(1-0) + 1 - 0\right)$$

$= 2\left(\frac{1}{5}(1-0) + (1-0)\right)$

$= 2\left(\frac{1}{5} + \frac{5}{5}\right)$

$= 2 \cdot \frac{6}{5} = \boxed{\frac{12}{5}}$

ex. $\int_{-\pi/4}^{\pi/4} \frac{\tan x}{x^2}\,dx = 0$ odd

Section 5.6 - Integration by Parts (start)

Often multiplied functions being integrated do not contain a useful derivative to allow for u-substitution (use of Substitution Rule)

Another integration tool is Integration by Parts (IBP)

one function will become "u" other function will be considered a derivative but not of u: "dv"

ex find $\int x\, e^x\,dx$ ($\int u\,dv$)

when pick which is "u" pick what will simplify (sometimes disappears) as a derivative

BUILD A CHART:

differentiate $u = x$, $du = dx$

$dv = e^x\,dx$ integrate $v = e^x$

$$\boxed{\text{IBP Formula: } \int u\,dv = uv - \int v\,du}$$

$\int x e^x\,dx = \int u\,dv = xe^x - \int e^x\,dx$

$= \boxed{xe^x - e^x + C}$

ex calculate $\int \underset{u}{x}\,\underset{dv}{e^{3x}\,dx}$

$u = x$, $du = dx$

$dv = e^{3x}\,dx$

$v = \int e^{3x}\,dx$

$= \frac{1}{3}\int e^u\,du$

$= \frac{1}{3}e^u$

$= \frac{1}{3}e^{3x}$

use u-sub:
$u = 3x$
$du = 3\,dx$
$\frac{1}{3}du = dx$

$$\int u \cdot dv = uv - \int v \cdot du$$

$= uv - \int v\,du$

$= x\left(\frac{1}{3}e^{3x}\right) - \frac{1}{3}\int e^{3x}\,dx$ now

$= \frac{1}{3}xe^{3x} - \frac{1}{3}\cdot\frac{1}{3}e^{3x} + C$

$= \boxed{\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C}$

ex. evaluate $\int x \sin x\,dx$

$= \int u\,dv$

$u = x$, $du = dx$

$dv = \sin x\,dx$

$v = \int \sin x\,dx$

$v = -\cos x$

$$\int u \cdot dv = uv - \int v \cdot du$$

$= \int u \, dv$ $\quad du = dx \quad \hookrightarrow v = \int \sin x \, dx$

$v = -\cos x$

$= uv - \int v \, du$

$= -x\cos x + \int \cos x \, dx$

$= \boxed{-x\cos x + \sin x + C}$